PERGAMON

International Journal of Solids and Structures 40 (2003) 5839–5860

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

The plane elasticity problem of an isotropic wedge under normal and shear distributed loading—application in the case of a multi-material problem

I.H. Stampouloglou, E.E. Theotokoglou *

Laboratory of Strength of Materials, Department of Mechanics, Faculty of Applied Sciences, The National Technical University of Athens, Zographou Campus, Theocaris, Bld, GR-157 73 Athens, Greece

Received 14 November 2002; received in revised form 24 April 2003

Abstract

The problem of a multi-material composite wedge under a normal and shear loading at its external faces is considered with a variable separable solution. The stress and displacement fields are determined using the equilibrium conditions for forces and moments and the appropriate Airy stress function. The infinite isotropic wedge under shear and normal distributed loading along its external faces is examined for different values of the order n of the radial coordinate r. The proposed solution is applied to the elastostatic problem of a composite isotropic k-materials infinite wedge under distributed loading along its external faces. Applications are made in the case of the two-materials composite wedge under linearly distributed loading along its external faces and in the case of a three-materials composite wedge under a parabolically distributed loading along its external faces.

Keywords: Plane elasticity; Isotropic; Wedge; Multi-material; Airy stress function

1. Introduction

The plane elasticity problem of a composite body consisting of a number of dissimilar wedges of arbitrary angles such that all interfaces coalesce at the same vertex O (Fig. 1) is in great interest in many engineering fields including automotive, microelectronics, aerospace, maritime, and nuclear engineering. In addition, when considering the strength of a composite, a very important part that has to be studied is the interface between two dissimilar materials. First Williams (1952) using the Airy stress function, developed the eigen-function expansion method in order to study the single-material wedge for several combinations of homogeneous boundary conditions. A lot of other investigations concerning material and geometric complications have followed by Bogy (1968, 1970), Dundurs (1969), Gdoutos and Theocaris (1975),

* Corresponding author. Address: Sofouli 33, Nea Smirni, Athens 17122, Greece.
E-mail address: stathis@central.ntua.gr (E.E. Theotokoglou).

doi:10.1016/S0020-7683(03)00296-8

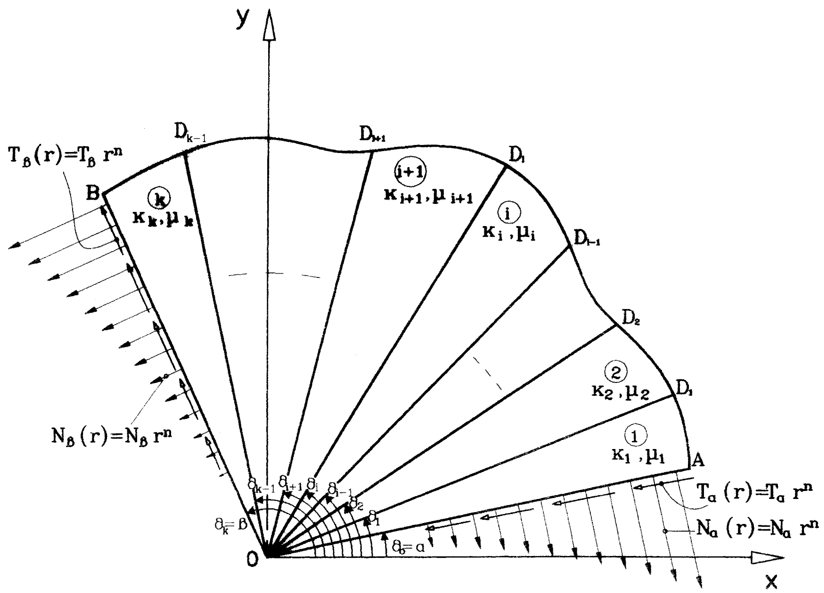

Fig. 1. Composite multi-material wedge under distributed loads at its faces.

Theocaris et al. (1979), Dempsey and Sinclair (1981), Ting (1984, 1996), Lau and Delale (1988), Dundurs and Markenscoff (1989), Barber (1992), Pageau et al. (1994), Joseph and Zhang (1998), Mantic et al. (1997), Horgan (1998) and Chen (1998).

Our study considers an infinite isotropic wedge under a polynomial-distributed loading (Fig. 1) at its external faces. It is also supposed that the distributed loading along the faces fulfills the self-similarity condition given by

$$N(\alpha r) = f_N(\alpha)N(r), \quad f_N(1) = 1, \quad \alpha \in R \tag{1}$$

in the case of a normal distributed loading; and

$$T(\alpha r) = f_T(\alpha)T(r), \quad f_T(1) = 1, \quad \alpha \in R \tag{2}$$

in the case of a shear distributed loading, where $f_N(\alpha)$ and $f_T(\alpha)$ are the similarity functions and $N(r)$, $T(r)$ are polynomials.

It is proved that the polynomials $N(r)$ and $T(r)$ in order to satisfy the self-similarity relations (1) and (2), must be mononyms of the form

$$\begin{aligned} N(r) &= Nr^n, \quad n = 0, \pm 1, \pm 2, \ldots \\ T(r) &= Tr^n, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{3}$$

where N, T constants, and n the order of the r.

The self-similarity property of the loading and the non-existence of a "characteristic dimension" in the geometry of the problem, enforces a variable-separable form of the stress field, thus

$$\sigma_{ij}(\alpha r, \theta) = f_{ij}(\alpha)\sigma_{ij}(r, \theta)$$

and for $\alpha = 1/r$, we have

$$\sigma_{ij}(r, \theta) = h_{ij}(r)g_{ij}(\theta), \quad i, j = r, \theta \tag{4}$$

where

$$h_{ij}(r) = \left[f_{ij}\left(\frac{1}{r}\right) \right]^{-1}, \quad g_{ij}(\theta) = \sigma_{ij}(1,\theta)$$

From relation (4), the distributed loads (3) along the faces $\theta = \alpha$ and $\theta = \beta$ of the wedge, take the form

(i) shear loading

$$\begin{aligned} T_\alpha(r) &= T_\alpha r^n = \sigma_{r\theta}(r, \theta = \alpha) \\ T_\beta(r) &= T_\beta r^n = \sigma_{r\theta}(r, \theta = \beta) \end{aligned} \tag{5}$$

thus

$$h_{r\theta} = r^n, \quad T_\alpha = g_{r\theta}(\theta = \alpha), \quad T_\beta = g_{r\theta}(\theta = \beta) \tag{6}$$

(ii) normal loading

$$\begin{aligned} N_\alpha(r) &= N_\alpha r^n = \sigma_{\theta\theta}(r, \theta = \alpha) \\ N_\beta(r) &= N_\beta r^n = \sigma_{\theta\theta}(r, \theta = \beta) \end{aligned} \tag{7}$$

thus

$$h_{\theta\theta}(r) = r^n, \quad N_\alpha = g_{\theta\theta}(\theta = \alpha), \quad N_\beta = g_{\theta\theta}(\theta = \beta) \tag{8}$$

Using the equilibrium conditions for forces and moments, and for different values of n, the unknown functions $h_{ij}(r)$ in the stress field expressions are determined. Selecting appropriate terms from the Michell tables (Michell, 1899; Barber, 1992), the Airy stress function, the $g_{ij}(\theta)$ functions and the stress fields are easily obtained. Finally applying the boundary conditions, the unknown coefficients of the stress fields are determined in the cases of shear distributed loading, normal distributed loading and uniformly distributed loading ($n = 0$).

The advantages of the proposed solution are

(i) The use of self-similarity property in the wedge elastostatic problem not only for concentrated loads at the apex but also for distributed loads along the faces of the wedge.
(ii) The determination of the stress function from the Michell tables (Michell, 1899) according to the required order of r because of the self-similarity property.

The contribution of our study relative to other investigations (Theocaris et al., 1979; Dempsey and Sinclair, 1981; Pageau et al., 1994; Ting, 1996; Mantic et al., 1997; Joseph and Zhang, 1998; Chen, 1998) is the solution of the elastostatic problem of a multi-material wedge and the determination of the stress and displacement fields not close to the singular point at the apex of the wedge, instead of just the determination of the order of singularity at the apex.

An analytical solution is proposed, based on the self-similarity property (variable-separable formulations). Using the superposition principle, the stress and displacement fields are determined analytically in the case of a multi-material wedge with different material properties under a polynomial distributed loading along the external faces of the wedge.

The proposed solution is applied to the elastostatic problem of an infinite multi-material isotropic wedge under distributed loading along its faces. Applications are made in the case of a composite two-materials infinite wedge under a linear distributed loading along its external faces and in the case of a three-material infinite isotropic wedge under a parabolic ($n = 2$) distributed loading along its external faces.

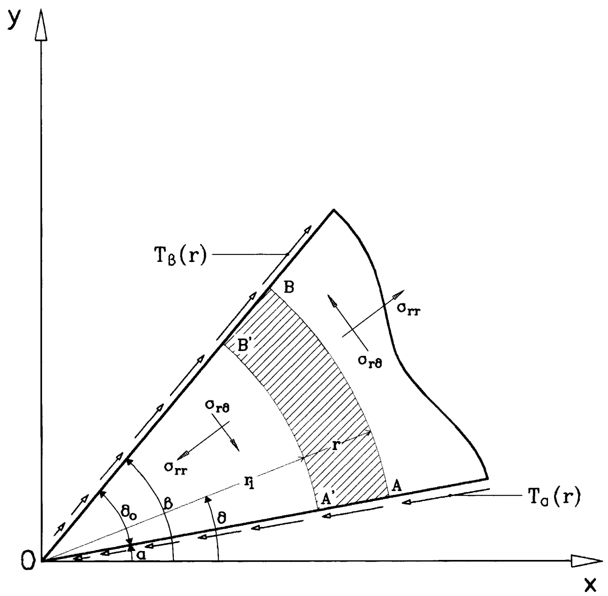

Fig. 2. Wedge under distributed shear loads at its faces.

2. Shear distributed loading

Using the equilibrium conditions for forces along the x–x and y–y axes (Fig. 2) for the element $(AA'B'B)$ and taking into account relations (5) and (6), it is obtained

$$h_{rr}(r) = h_{r\theta}(r) = r^n \tag{9}$$

$$\begin{aligned} \int_\alpha^\beta g_{rr}(\theta)\cos\theta\,\mathrm{d}\theta &= \frac{T_\alpha\cos\alpha - T_\beta\cos\beta}{n+1} + \int_\alpha^\beta g_{r\theta}(\theta)\sin\theta\,\mathrm{d}\theta, \quad n \neq -1 \\ \int_\alpha^\beta g_{rr}(\theta)\sin\theta\,\mathrm{d}\theta &= \frac{T_\alpha\sin\alpha - T_\beta\sin\beta}{n+1} - \int_\alpha^\beta g_{r\theta}(\theta)\cos\theta\,\mathrm{d}\theta, \quad n \neq -1 \end{aligned} \tag{10}$$

From the equilibrium of moments at the element $(AA'B'B)$, it is also obtained

$$\int_\alpha^\beta g_{r\theta}(\theta)\,\mathrm{d}\theta = 0 \tag{11}$$

From the Michell tables (Michell, 1899; Barber, 1992) we can select terms in order to formulate the Airy stress function which ensures the required order of r. Because of the use of the Michell tables we distinguish the following cases.

2.1. The case $n = -2$

From relations (5), (6) and (9), it is obtained

$$T_\alpha(r) = \frac{T_\alpha}{r^2}, \quad T_\beta(r) = \frac{T_\beta}{r^2}, \quad h_{rr}(r) = h_{r\theta}(r) = \frac{1}{r^2} \tag{12}$$

The corresponding Airy stress function is

$$M(r,\theta) = \Gamma_{04}\ln r + \Gamma'_{03}\theta + \Gamma_{23}\cos(2\theta) + \Gamma'_{23}\sin(2\theta) \tag{13}$$

The boundary conditions of the problem are

$$\begin{aligned} &\theta = \alpha = -\frac{\theta_0}{2}; \quad \sigma_{r\theta}\left(r, -\frac{\theta_0}{2}\right) = \frac{T_\alpha}{r^2}, \quad \sigma_{\theta\theta}\left(r, -\frac{\theta_0}{2}\right) = 0 \\ &\theta = \beta = \frac{\theta_0}{2}; \quad \sigma_{r\theta}\left(r, \frac{\theta_0}{2}\right) = \frac{T_\beta}{r^2}, \quad \sigma_{\theta\theta}\left(r, \frac{\theta_0}{2}\right) = 0 \end{aligned} \tag{14}$$

The stress and the displacement fields satisfying the stress function (13), are

$$\begin{aligned} \sigma_{rr}(r,\theta) &= \frac{\Gamma_{04}}{r^2} - \frac{4\Gamma_{23}}{r^2}\cos(2\theta) - \frac{4\Gamma'_{23}}{r^2}\sin(2\theta) \\ \sigma_{r\theta}(r,\theta) &= \frac{\Gamma'_{03}}{r^2} - \frac{2\Gamma_{23}}{r^2}\sin(2\theta) + \frac{2\Gamma'_{23}}{r^2}\cos(2\theta) \\ \sigma_{\theta\theta}(r,\theta) &= -\frac{\Gamma_{04}}{r^2} \end{aligned} \tag{15}$$

and

$$\begin{aligned} 2\mu u_r(r,\theta) &= -\frac{\Gamma_{04}}{r} + \frac{(\kappa+1)\Gamma_{23}}{r}\cos(2\theta) + \frac{(\kappa+1)\Gamma'_{23}}{r}\sin(2\theta) \\ 2\mu u_\theta(r,\theta) &= -\frac{\Gamma'_{03}}{r} - \frac{(\kappa-1)\Gamma_{23}}{r}\sin(2\theta) + \frac{(\kappa-1)\Gamma'_{23}}{r}\cos(2\theta) \end{aligned} \tag{16}$$

where μ is the shear modulus, $\kappa = (3-4\nu)$ for plane strain, $\kappa = (3-\nu)/(1+\nu)$ for generalized plane stress, ν being the Poisson's ratio; while the unknown coefficients Γ_{04}, Γ'_{03}, Γ_{23}, Γ'_{23} derived from the boundary conditions (14) and the equilibrium of moments (11), are

$$\begin{aligned} &\Gamma_{04} = 0, \quad \Gamma'_{03} = -\frac{(T_\alpha + T_\beta)\tan\theta_0}{2(\theta_0 - \tan\theta_0)} \\ &\Gamma_{23} = \frac{T_\alpha - T_\beta}{4\sin\theta_0}, \quad \Gamma'_{23} = \frac{(T_\alpha + T_\beta)\theta_0}{4\cos\theta_0(\theta_0 - \tan\theta_0)} \end{aligned} \tag{17}$$

where

$$\sin(2\theta_0) \neq 0, \quad \theta_0 - \tan\theta_0 \neq 0, \quad \theta_0 = \beta - \alpha.$$

Using relations (15) for the functions $g_{ij}(\theta)$ $(i,j = r,\theta)$ and relations (17), relations (10) are satisfied.

2.2. The case $n = -m \leqslant -3$

From relations (5), (6) and (9), it is obtained

$$T_\alpha(r) = \frac{T_\alpha}{r^m}, \quad T_\beta(r) = \frac{T_\beta}{r^m}, \quad h_{rr}(r) = h_{r\theta}(r) = \frac{1}{r^m} \tag{18}$$

The corresponding Airy stress function is

$$M(r,\theta) = \Gamma_{m3}\frac{\cos(m\theta)}{r^{m-2}} + \Gamma_{(m-2)4}\frac{\cos[(m-2)\theta]}{r^{m-2}} + \Gamma'_{m3}\frac{\sin(m\theta)}{r^{m-2}} + \Gamma'_{(m-2)4}\frac{\sin[(m-2)\theta]}{r^{m-2}}, \quad m = -n \geqslant 3 \tag{19}$$

The boundary conditions of the problem are

$$\begin{aligned} &\theta = \alpha = -\frac{\theta_0}{2}; \quad \sigma_{r\theta}\left(r, -\frac{\theta_0}{2}\right) = \frac{T_\alpha}{r^m}, \quad \sigma_{\theta\theta}\left(r, -\frac{\theta_0}{2}\right) = 0 \\ &\theta = \beta = \frac{\theta_0}{2}; \quad \sigma_{r\theta}\left(r, \frac{\theta_0}{2}\right) = \frac{T_\beta}{r^m}, \quad \sigma_{\theta\theta}\left(r, \frac{\theta_0}{2}\right) = 0 \end{aligned} \tag{20}$$

The stress and the displacement fields satisfying the stress function (19), are

$$\begin{aligned} \sigma_{rr}(r,\theta) = & -\Gamma_{m3}(m+2)(m-1)\frac{\cos(m\theta)}{r^m} - \Gamma_{(m-2)4}(m-2)(m-1)\frac{\cos[(m-2)\theta]}{r^m} \\ & - \Gamma'_{m3}(m+2)(m-1)\frac{\sin(m\theta)}{r^m} - \Gamma'_{(m-2)4}(m-2)(m-1)\frac{\sin[(m-2)\theta]}{r^m} \\ \sigma_{r\theta}(r,\theta) = & -\Gamma_{m3}m(m-1)\frac{\sin(m\theta)}{r^m} - \Gamma_{(m-2)4}(m-2)(m-1)\frac{\sin[(m-2)\theta]}{r^m} \\ & + \Gamma'_{m3}m(m-1)\frac{\cos(m\theta)}{r^m} + \Gamma'_{(m-2)4}(m-2)(m-1)\frac{\cos[(m-2)\theta]}{r^m} \\ \sigma_{\theta\theta}(r,\theta) = & \ \Gamma_{m3}(m-1)(m-2)\frac{\cos(m\theta)}{r^m} + \Gamma_{(m-2)4}(m-2)(m-1)\frac{\cos[(m-2)\theta]}{r^m} \\ & + \Gamma'_{m3}(m-1)(m-2)\frac{\sin(m\theta)}{r^m} + \Gamma'_{(m-2)4}(m-2)(m-1)\frac{\sin[(m-2)\theta]}{r^m} \end{aligned} \tag{21}$$

and

$$\begin{aligned} 2\mu u_r(r,\theta) = & \frac{1}{r^{m-1}}[\Gamma_{m3}(\kappa+m-1)\cos(m\theta) + \Gamma_{(m-2)4}(m-2)\cos[(m-2)\theta] + \Gamma'_{m3}(\kappa+m-1)\sin(m\theta) \\ & + \Gamma'_{(m-2)4}(m-2)\sin[(m-2)\theta]] \\ 2\mu u_\theta(r,\theta) = & \frac{1}{r^{m-1}}[-\Gamma_{m3}(\kappa-m+1)\sin(m\theta) + \Gamma_{(m-2)4}(m-2)\sin[(m-2)\theta] + \Gamma'_{m3}(\kappa-m+1)\cos(m\theta) \\ & - \Gamma'_{(m-2)4}(m-2)\cos[(m-2)\theta]] \end{aligned} \tag{22}$$

where the unknown coefficients Γ_{m3}, Γ'_{m3}, $\Gamma_{(m-2)4}$, $\Gamma'_{(m-2)4}$ are derived from the boundary conditions (20) using (21). Thus

$$\begin{aligned} \Gamma_{m3} &= \frac{(T_\alpha - T_\beta)\cos\left(m\frac{\theta_0}{2} - \theta_0\right)}{2(m-1)((m-1)\sin\theta_0 + \sin[(m-1)\theta_0])} \\ \Gamma'_{m3} &= \frac{(T_\alpha + T_\beta)\sin\left(m\frac{\theta_0}{2} - \theta_0\right)}{2(m-1)(\sin[(m-1)\theta_0] - (m-1)\sin\theta_0)} \\ \Gamma_{(m-2)4} &= -\frac{(T_\alpha - T_\beta)\cos\left(m\frac{\theta_0}{2}\right)}{2(m-1)((m-1)\sin\theta_0 + \sin[(m-1)\theta_0])} \\ \Gamma'_{(m-2)4} &= -\frac{(T_\alpha + T_\beta)\sin\left(m\frac{\theta_0}{2}\right)}{2(m-1)(\sin[(m-1)\theta_0] - (m-1)\sin\theta_0)} \end{aligned} \tag{23}$$

where

$$\sin[(m-1)\theta_0] \neq \pm(m-1)\sin\theta_0, \quad \theta_0 = \beta - \alpha$$

Using relations (21) for the functions $g_{ij}(\theta)$ $(i,j = r,\theta)$ and relations (23), the equilibrium conditions (10) and (11), are satisfied.

2.3. The case $n \geqslant 1$

From relations (5), (6) and (9), it is obtained

$$T(r) = T_a r^n, \quad T_\beta(r) = T_\beta r^n, \quad h_{rr}(r) = h_{r\theta}(r) = r^n \tag{24}$$

The corresponding Airy stress function is

$$M(r,\theta) = \Gamma_{n1} r^{n+2}\cos(n\theta) + \Gamma'_{n1} r^{n+2}\sin(n\theta) + \Gamma_{(n+2)2} r^{n+2}\cos[(n+2)\theta] + \Gamma'_{(n+2)2} r^{n+2}\sin[(n+2)\theta] \tag{25}$$

The boundary conditions of the problem are

$$\begin{aligned} &\theta = \alpha = -\frac{\theta_0}{2}; \quad \sigma_{r\theta}\left(r, -\frac{\theta_0}{2}\right) = T_a r^n, \quad \sigma_{\theta\theta}\left(r, -\frac{\theta_0}{2}\right) = 0 \\ &\theta = \beta = \frac{\theta_0}{2}; \quad \sigma_{r\theta}\left(r, \frac{\theta_0}{2}\right) = T_\beta r^n, \quad \sigma_{\theta\theta}\left(r, \frac{\theta_0}{2}\right) = 0 \end{aligned} \tag{26}$$

Correspondingly the stress and the displacement fields, are

$$\begin{aligned} \sigma_{rr}(r,\theta) = & -\Gamma_{n1}(n+1)(n-2)r^n\cos(n\theta) - \Gamma'_{n1}(n+1)(n-2)r^n\sin(n\theta) \\ & - \Gamma_{(n+2)2}(n+2)(n+1)r^n\cos[(n+2)\theta] - \Gamma'_{(n+2)2}(n+2)(n+1)r^n\sin[(n+2)\theta] \\ \sigma_{r\theta}(r,\theta) = & \Gamma_{n1} n(n+1)r^n\sin(n\theta) - \Gamma'_{n1} n(n+1)r^n\cos(n\theta) + \Gamma_{(n+2)2}(n+2)(n+1)r^n\sin[(n+2)\theta] \\ & - \Gamma'_{(n+2)2}(n+2)(n+1)r^n\cos[(n+2)\theta] \\ \sigma_{\theta\theta}(r,\theta) = & \Gamma_{n1}(n+1)(n+2)r^n\cos(n\theta) + \Gamma'_{n1}(n+1)(n+2)r^n\sin(n\theta) \\ & + \Gamma_{(n+2)2}(n+2)(n+1)r^n\cos[(n+2)\theta] + \Gamma'_{(n+2)2}(n+2)(n+1)r^n\sin[(n+2)\theta] \end{aligned} \tag{27}$$

and

$$\begin{aligned} 2\mu u_r(r,\theta) = & (\kappa - n - 1)r^{n+1}[\Gamma_{n1}\cos(n\theta) + \Gamma'_{n1}\sin(n\theta)] \\ & - (n+2)r^{n+1}\{\Gamma_{(n+2)2}\cos[(n+2)\theta] + \Gamma'_{(n+2)2}\sin[(n+2)\theta]\} \\ 2\mu u_\theta(r,\theta) = & (\kappa + n + 1)r^{n+1}[\Gamma_{n1}\sin(n\theta) - \Gamma'_{n1}\cos(n\theta)] \\ & + (n+2)r^{n+1}\{\Gamma_{(n+2)2}\sin[(n+2)\theta] - \Gamma'_{(n+2)2}\cos[(n+2)\theta]\} \end{aligned} \tag{28}$$

Using the boundary conditions (26) and the stress field (27), the unknown coefficients Γ_{n1}, Γ'_{n1}, $\Gamma_{(n+2)2}$ and $\Gamma'_{(n+2)2}$, are

$$\begin{aligned} \Gamma_{n1} &= \frac{(T_\alpha - T_\beta)\cos\left[(n+2)\frac{\theta_0}{2}\right]}{2(n+1)(\sin[(n+1)\theta_0] + (n+1)\sin\theta_0)} \\ \Gamma'_{n1} &= \frac{(T_\alpha + T_\beta)\sin\left[(n+2)\frac{\theta_0}{2}\right]}{2(n+1)(\sin[(n+1)\theta_0] - (n+1)\sin\theta_0)} \\ \Gamma_{(n+2)2} &= -\frac{(T_\alpha - T_\beta)\cos\left(n\frac{\theta_0}{2}\right)}{2(n+1)(\sin[(n+1)\theta_0] + (n+1)\sin\theta_0)} \\ \Gamma'_{(n+2)2} &= -\frac{(T_\alpha + T_\beta)\sin\left(n\frac{\theta_0}{2}\right)}{2(n+1)(\sin[(n+1)\theta_0] - (n+1)\sin\theta_0)} \end{aligned} \tag{29}$$

where

$$\sin[(n+1)\theta_0] \neq \pm(n+1)\sin\theta_0, \quad \theta_0 = \beta - \alpha$$

The equilibrium conditions (10) and (11) are satisfied using relations (27) and (29).

3. Normal distributed loading

From the equilibrium of moments at the element $(AA'B'B)$ (Fig. 3), it is obtained

$$h_{r\theta}(r) = r^n \tag{30}$$

$$\int_{\alpha}^{\beta} g_{r\theta}(\theta)\,\mathrm{d}\theta = \frac{N_\alpha - N_\beta}{n+2}, \quad n \neq -2 \tag{31}$$

In a similar way from the equilibrium of forces along the x–x and y–y axes for the element $(AA'B'B)$ (Fig. 3) and taking into consideration relations (7) and (8), we have

$$h_{rr}(r) = h_{\theta\theta}(r) = r^n \tag{32}$$

$$\begin{aligned} \int_{\alpha}^{\beta} g_{rr}(\theta)\cos\theta\,\mathrm{d}\theta &= -\frac{N_\alpha \sin\alpha - N_\beta \sin\beta}{n+1} + \int_{\alpha}^{\beta} g_{r\theta}(\theta)\sin\theta\,\mathrm{d}\theta, \quad n \neq -1 \\ \int_{\alpha}^{\beta} g_{rr}(\theta)\sin\theta\,\mathrm{d}\theta &= \frac{N_\alpha \cos\alpha - N_\beta \cos\beta}{n+1} - \int_{\alpha}^{\beta} g_{r\theta}(\theta)\cos\theta\,\mathrm{d}\theta, \quad n \neq -1 \end{aligned} \tag{33}$$

Using the Michell tables as in Section 2, we distinguish the following cases.

3.1. The case $n = -m \leqslant -3$

From relations (7), (8), (30) and (32), it is obtained

$$N_\alpha(r) = \frac{N_\alpha}{r^m}, \quad N_\beta(r) = \frac{N_\beta}{r^m}, \quad h_{rr}(r) = h_{r\theta}(r) = h_{\theta\theta}(r) = \frac{1}{r^m} \tag{34}$$

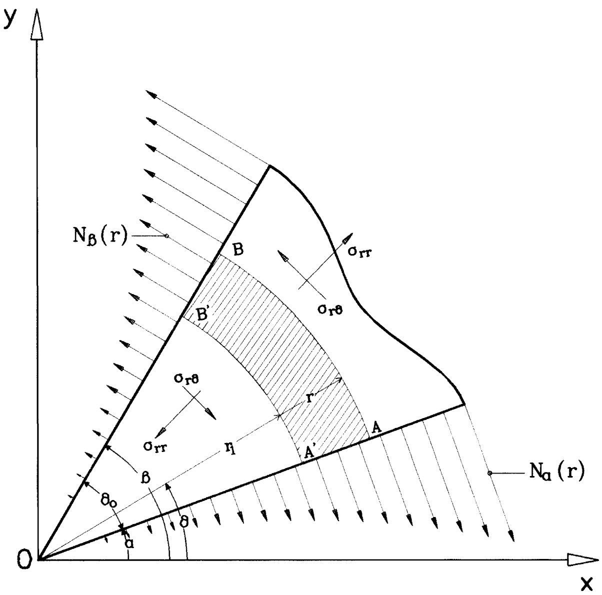

Fig. 3. Wedge under distributed normal loads at its faces.

The Airy stress function ensuring the required order of r, is

$$M(r,\theta) = \Gamma_{m3}\frac{\cos(m\theta)}{r^{m-2}} + \Gamma_{(m-2)4}\frac{\cos[(m-2)\theta]}{r^{m-2}} + \Gamma'_{m3}\frac{\sin(m\theta)}{r^{m-2}} + \Gamma'_{(m-2)4}\frac{\sin[(m-2)\theta]}{r^{m-2}},\quad n = -m \leqslant -3 \tag{35}$$

The boundary conditions are

$$\begin{aligned}&\theta = \alpha = -\frac{\theta_0}{2};\quad \sigma_{r\theta}\left(r,-\frac{\theta_0}{2}\right) = 0,\quad \sigma_{\theta\theta}\left(r,-\frac{\theta_0}{2}\right) = \frac{N_\alpha}{r^m}\\&\theta = \beta = \frac{\theta_0}{2};\quad \sigma_{r\theta}\left(r,\frac{\theta_0}{2}\right) = 0,\quad \sigma_{\theta\theta}\left(r,\frac{\theta_0}{2}\right) = \frac{N_\beta}{r^m}\end{aligned} \tag{36}$$

The stress and displacement fields derived from the stress function (35), are

$$\begin{aligned}\sigma_{rr}(r,\theta) =& -(m+2)(m-1)\Gamma_{m3}\frac{\cos(m\theta)}{r^m} - (m-2)(m-1)\Gamma_{(m-2)4}\frac{\cos[(m-2)\theta]}{r^m}\\&-(m+2)(m-1)\Gamma'_{m3}\frac{\sin(m\theta)}{r^m} - (m-2)(m-1)\Gamma'_{(m-2)4}\frac{\sin[(m-2)\theta]}{r^m}\\\sigma_{r\theta}(r,\theta) =& -m(m-1)\Gamma_{m3}\frac{\sin(m\theta)}{r^m} - (m-2)(m-1)\Gamma_{(m-2)4}\frac{\sin[(m-2)\theta]}{r^m}\\&+m(m-1)\Gamma'_{m3}\frac{\cos(m\theta)}{r^m} + (m-2)(m-1)\Gamma'_{(m-2)4}\frac{\cos[(m-2)\theta]}{r^m}\\\sigma_{\theta\theta}(r,\theta) =& (m-1)(m-2)\Gamma_{m3}\frac{\cos(m\theta)}{r^m} + (m-1)(m-2)\Gamma_{(m-2)4}\frac{\cos[(m-2)\theta]}{r^m}\\&+(m-1)(m-2)\Gamma'_{m3}\frac{\sin(m\theta)}{r^m} + (m-1)(m-2)\Gamma'_{(m-2)4}\frac{\sin[(m-2)\theta]}{r^m}\end{aligned} \tag{37}$$

and

$$\begin{aligned}2\mu u_r(r,\theta) =& \frac{1}{r^{m-1}}[\Gamma_{m3}(\kappa+m-1)\cos(m\theta) + \Gamma_{(m-2)4}(m-2)\cos[(m-2)\theta] + \Gamma'_{m3}(\kappa+m-1)\sin(m\theta)\\&+\Gamma'_{(m-2)4}(m-2)\sin[(m-2)\theta]]\\2\mu u_\theta(r,\theta) =& \frac{1}{r^{m-1}}[-\Gamma_{m3}(\kappa-m+1)\sin(m\theta) + \Gamma_{(m-2)4}(m-2)\sin[(m-2)\theta] + \Gamma'_{m3}(\kappa-m-1)\cos(m\theta)\\&-\Gamma'_{(m-2)4}(m-2)\cos[(m-2)\theta]]\end{aligned} \tag{38}$$

where the unknown coefficients Γ_{m3}, Γ'_{m3}, $\Gamma_{(m-2)4}$, $\Gamma'_{(m-2)4}$ are calculated using the boundary conditions (36) and the relations (37), thus

$$\begin{aligned}
\Gamma_{m3} &= -\frac{(N_\alpha + N_\beta)\sin\left[(m-2)\frac{\theta_0}{2}\right]}{2(m-1)((m-1)\sin\theta_0 + \sin[(m-1)\theta_0])}\\
\Gamma'_{m3} &= -\frac{(N_\alpha - N_\beta)\cos\left[(m-2)\frac{\theta_0}{2}\right]}{2(m-1)((m-1)\sin\theta_0 - \sin[(m-1)\theta_0])}\\
\Gamma_{(m-2)4} &= \frac{m(N_\alpha + N_\beta)\sin\left(m\frac{\theta_0}{2}\right)}{2(m-1)(m-2)((m-1)\sin\theta_0 + \sin[(m-1)\theta_0])}\\
\Gamma'_{(m-2)4} &= \frac{m(N_\alpha - N_\beta)\cos\left(m\frac{\theta_0}{2}\right)}{2(m-1)(m-2)((m-1)\sin\theta_0 - \sin[(m-1)\theta_0])}
\end{aligned} \tag{39}$$

where

$$\sin[(m-1)\theta_0] \neq \pm(m-1)\sin\theta_0, \quad \theta_0 = \beta - \alpha.$$

Using relations (37) for the functions $g_{ij}(\theta)$ $(i, j = r, \theta)$ and relations (39), the equilibrium conditions (31) and (33) are satisfied.

3.2. The case $n \geqslant 1$

From relations (7), (8), (30) and (32), we have

$$N_\alpha(r) = N_\alpha r^n, \quad N_\beta(r) = N_\beta r^n, \quad h_{rr}(r) = h_{r\theta}(r) = h_{\theta\theta}(r) = r^n \tag{40}$$

The Airy stress function ensuring the order of r, is

$$M(r,\theta) = \Gamma_{n1} r^{n+2}\cos(n\theta) + \Gamma'_{n1} r^{n+2}\sin(n\theta) + \Gamma_{(n+2)2} r^{n+2}\cos[(n+2)\theta] + \Gamma'_{(n+2)2} r^{n+2}\sin[(n+2)\theta] \tag{41}$$

The boundary conditions are

$$\begin{aligned}
&\theta = \alpha = -\frac{\theta_0}{2}; \quad \sigma_{r\theta}\left(r, -\frac{\theta_0}{2}\right) = 0, \quad \sigma_{\theta\theta}\left(r, -\frac{\theta_0}{2}\right) = N_\alpha r^n\\
&\theta = \beta = \frac{\theta_0}{2}; \quad \sigma_{r\theta}\left(r, \frac{\theta_0}{2}\right) = 0, \quad \sigma_{\theta\theta}\left(r, \frac{\theta_0}{2}\right) = N_\beta r^n
\end{aligned} \tag{42}$$

The stress and displacement fields obtained from the stress function (41), are

$$\begin{aligned}
\sigma_{rr}(r,\theta) = &-(n+1)(n-2)\Gamma_{n1} r^n\cos(n\theta) - (n+1)(n-2)\Gamma'_{n1} r^n\sin(n\theta)\\
&-(n+1)(n+2)\Gamma_{(n+2)2} r^n\cos[(n+2)\theta] - (n+1)(n+2)\Gamma'_{(n+2)2} r^n\sin[(n+2)\theta]\\
\sigma_{r\theta}(r,\theta) = &\; n(n+1)\Gamma_{n1} r^n\sin(n\theta) - n(n+1)\Gamma'_{n1} r^n\cos(n\theta) + (n+1)(n+2)\Gamma_{(n+2)2} r^n\sin[(n+2)\theta]\\
&-(n+1)(n+2)\Gamma'_{(n+2)2} r^n\cos[(n+2)\theta]\\
\sigma_{\theta\theta}(r,\theta) = &\; (n+1)(n+2)\Gamma_{n1} r^n\cos(n\theta) + (n+1)(n+2)\Gamma'_{n1} r^n\sin(n\theta)\\
&+(n+1)(n+2)\Gamma_{(n+2)2} r^n\cos[(n+2)\theta] + (n+1)(n+2)\Gamma'_{(n+2)2} r^n\sin[(n+2)\theta]
\end{aligned} \tag{43}$$

and

$$\begin{aligned} 2\mu u_r(r,\theta) &= r^{n+1}\Big\{(\kappa-n-1)\Big[\Gamma_{n1}\cos(n\theta)+\Gamma'_{n1}\sin(n\theta)\Big] \\ &\quad -(n+2)\Big[\Gamma_{(n+2)2}\cos[(n+2)\theta]+\Gamma'_{(n+2)2}\sin[(n+2)\theta]\Big]\Big\} \\ 2\mu u_\theta(r,\theta) &= r^{n+1}\Big\{(\kappa+n+1)\Big[\Gamma_{n1}\sin(n\theta)-\Gamma'_{n1}\cos(n\theta)\Big] \\ &\quad +(n+2)\Big[\Gamma_{(n+2)2}\sin[(n+2)\theta]-\Gamma'_{(n+2)2}\cos[(n+2)\theta]\Big]\Big\} \end{aligned} \tag{44}$$

where the unknown coefficients Γ_{n1}, Γ'_{n1}, $\Gamma_{(n+2)2}$ and $\Gamma'_{(n+2)2}$ are obtained from the boundary conditions (42) and the stress field (43), thus

$$\begin{aligned} \Gamma_{n1} &= \frac{(N_\alpha+N_\beta)\sin\left[(n+2)\frac{\theta_0}{2}\right]}{2(n+1)(\sin[(n+1)\theta_0]+(n+1)\sin\theta_0)} \\ \Gamma'_{n1} &= -\frac{(N_\alpha-N_\beta)\cos\left[(n+2)\frac{\theta_0}{2}\right]}{2(n+1)(\sin[(n+1)\theta_0]-(n+1)\sin\theta_0)} \\ \Gamma_{(n+2)2} &= -\frac{n(N_\alpha+N_\beta)\sin\left(n\frac{\theta_0}{2}\right)}{2(n+1)(n+2)(\sin[(n+1)\theta_0]+(n+1)\sin\theta_0)} \\ \Gamma'_{(n+2)2} &= \frac{n(N_\alpha-N_\beta)\cos\left(n\frac{\theta_0}{2}\right)}{2(n+1)(n+2)(\sin[(n+1)\theta_0]-(n+1)\sin\theta_0)} \end{aligned} \tag{45}$$

where

$$\sin[(n+1)\theta_0] \neq \pm(n+1)\sin\theta_0, \quad \theta_0=\beta-\alpha.$$

Using relations (43) for the functions $g_{ij}(\theta)$ $(i,j=r,\theta)$ and relations (45), the equilibrium conditions (31) and (33) are satisfied.

4. Uniformly distributed loading (*n* = 0)

In the case of uniformly distributed loading (Fig. 4), relations (5) and (7) take the form

$$\begin{aligned} T_\alpha(r) &= T_\alpha = \sigma_{r\theta}(r,\alpha); \quad N_\alpha(r)=N_\alpha=\sigma_{\theta\theta}(r,\alpha) \\ T_\beta(r) &= T_\beta = \sigma_{r\theta}(r,\beta); \quad N_\beta(r)=N_\beta=\sigma_{\theta\theta}(r,\beta) \end{aligned} \tag{46}$$

From the equilibrium of moments at the element (OAB) (Fig. 4), we have

$$\sigma_{r\theta}(r,\theta)=g_{r\theta}(\theta) \tag{47}$$

$$\int_\alpha^\beta g_{r\theta}(\theta)\,\mathrm{d}\theta=\frac{1}{2}(N_\alpha-N_\beta) \tag{48}$$

Using the equilibrium of forces along the *x–x* and *y–y* axes (Fig. 4) for the element (OAB), it is obtained

$$\sigma_{rr}(r,\theta)=g_{rr}(\theta) \tag{49}$$

$$\begin{aligned} \int_\alpha^\beta g_{rr}(\theta)\cos\theta\,\mathrm{d}\theta-\int_\alpha^\beta g_{r\theta}(\theta)\sin\theta\,\mathrm{d}\theta &= -\big(N_\alpha\sin\alpha-N_\beta\sin\beta\big)+\big(T_\alpha\cos\alpha-T_\beta\cos\beta\big) \\ \int_\alpha^\beta g_{rr}(\theta)\sin\theta\,\mathrm{d}\theta+\int_\alpha^\beta g_{r\theta}(\theta)\cos\theta\,\mathrm{d}\theta &= \big(N_\alpha\cos\alpha-N_\beta\cos\beta\big)+\big(T_\alpha\sin\alpha-T_\beta\sin\beta\big) \end{aligned} \tag{50}$$

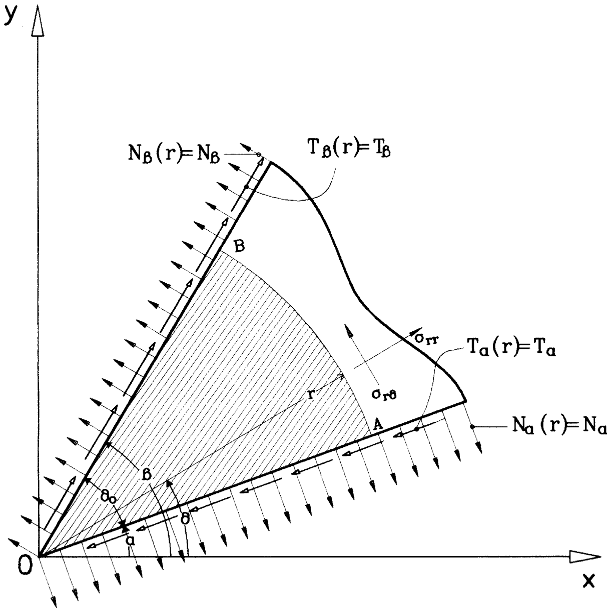

Fig. 4. Wedge under uniformly distributed loads at its faces.

On the other hand the equilibrium of moments (Fig. 4), gives

$$r^2\int_\alpha^\theta g_{r\theta}(\omega)\,\mathrm{d}\omega+\int_0^r\sigma_{\theta\theta}(s,\theta)s\,\mathrm{d}s-\int_0^r N_\alpha s\,\mathrm{d}s=0,\quad 0<s\leqslant r,\ \ a\leqslant\omega\leqslant\theta$$

The above relation, taking into account that

$$r^2=2\int_0^r s\,\mathrm{d}s$$

becomes

$$\sigma_{\theta\theta}(r,\theta)=g_{\theta\theta}(\theta)=N_\alpha-2G(\theta),\quad G(\theta)=\int_\alpha^\theta g_{r\theta}(\omega)\,\mathrm{d}\omega \tag{51}$$

From relations (47), (49) and (51), it follows that the stresses σ_{ij} are only depended on the angle θ, thus

$$\sigma_{ij}(r,\theta)=g_{ij}(\theta),\quad i,j=r,\theta \tag{52}$$

The Airy stress function ensuring a stress field independent of r (Barber, 1992), is

$$M(r,\theta)=\Gamma_{01}r^2+\Gamma'_{01}r^2\theta+\Gamma_{22}r^2\cos(2\theta)+\Gamma'_{22}r^2\sin(2\theta) \tag{53}$$

The boundary conditions are

$$\begin{aligned}&\theta=\alpha=-\frac{\theta_0}{2};\quad \sigma_{r\theta}\left(r,-\frac{\theta_0}{2}\right)=T_\alpha,\quad \sigma_{\theta\theta}\left(r,-\frac{\theta_0}{2}\right)=N_\alpha\\&\theta=\beta=\frac{\theta_0}{2};\quad \sigma_{r\theta}\left(r,\frac{\theta_0}{2}\right)=T_\beta,\quad \sigma_{\theta\theta}\left(r,\frac{\theta_0}{2}\right)=N_\beta\end{aligned} \tag{54}$$

The stress and displacement fields obtained from the stress function (53), are

$$\begin{aligned}
\sigma_{rr}(r,\theta) &= g_{rr}(\theta) = 2\Gamma_{01} + 2\Gamma'_{01}\theta - 2\Gamma_{22}\cos(2\theta) - 2\Gamma'_{22}\sin(2\theta) \\
\sigma_{r\theta}(r,\theta) &= g_{r\theta}(\theta) = -\Gamma'_{01} + 2\Gamma_{22}\sin(2\theta) - 2\Gamma'_{22}\cos(2\theta) \\
\sigma_{\theta\theta}(r,\theta) &= g_{\theta\theta}(\theta) = 2\Gamma_{01} + 2\Gamma'_{01}\theta + 2\Gamma_{22}\cos(2\theta) + 2\Gamma'_{22}\sin(2\theta)
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
2\mu u_r(r,\theta) &= r\Big[(\kappa-1)\Gamma_{01} + (\kappa-1)\theta\Gamma'_{01} - 2\Gamma_{22}\cos 2\theta - 2\Gamma'_{22}\sin 2\theta\Big] \\
2\mu u_\theta(r,\theta) &= r\Big[-2\Gamma'_{01}\ln r + 2\Gamma_{22}\sin 2\theta - 2\Gamma'_{22}\cos 2\theta\Big]
\end{aligned} \tag{56}$$

where the unknown coefficients Γ_{01}, Γ'_{01}, Γ_{22} and Γ'_{22} are obtained from the boundary conditions (54) and the stress field (55); hence

$$\begin{aligned}
\Gamma_{01} &= \frac{(N_\alpha + N_\beta)\sin\theta_0 + (T_\alpha - T_\beta)\cos\theta_0}{4\sin\theta_0} \\
\Gamma'_{01} &= \frac{(T_\alpha + T_\beta)\sin\theta_0 - (N_\alpha - N_\beta)\cos\theta_0}{2\cos\theta_0(\theta_0 - \tan\theta_0)} \\
\Gamma_{22} &= -\frac{T_\alpha - T_\beta}{4\sin\theta_0} \\
\Gamma'_{22} &= -\frac{(T_\alpha + T_\beta)\theta_0 - (N_\alpha - N_\beta)}{4\cos\theta_0(\theta_0 - \tan\theta_0)}
\end{aligned} \tag{57}$$

where

$$\theta_0 \neq \pi \text{ or } 2\pi \quad \text{and} \quad \theta_0 \neq \tan\theta_0 \ (\theta_0 \neq 1.43\pi).$$

Using relations (55) and (57), the equilibrium conditions (48), (50) and (51) are fulfilled.

5. Multi-material isotropic wedge under distributed loading

This section investigates the conditions in linear elasticity under which the compatibility of displacements among the interfaces of a multi-material wedge is succeeded, in order to ensure the variable separable solution of the stress field.

In the previous analysis for the equilibrium of moments and forces, the mechanical properties of materials do not appear because of the continuity of the displacement field. Taking into consideration the continuity, the compatibility of displacements and the self-similarity property, relation (4) is also valid in the case of a multi-material isotropic wedge, so

$$h_{rr}(r) = h_{r\theta}(r) = h_{\theta\theta}(r) = r^n \tag{58}$$

The solution of the multi-material wedge problem (Fig. 1) is studied for every sub-wedge considering that at the interfaces OD_i $(1 \leqslant i \leqslant k-1)$ act distributed normal and shear stresses with the same order of r, fulfilling the self-similarity property

$$\begin{aligned}
N_i(r) &= \sigma_{\theta\theta}(r,\theta=\theta_i) = h_{\theta\theta}(r)g_{\theta\theta}(\theta_i) = N_i r^n, \quad N_i = g_{\theta\theta}(\theta_i), \ 1 \leqslant i \leqslant k-1 \\
T_i(r) &= \sigma_{r\theta}(r,\theta=\theta_i) = h_{r\theta}(r)g_{r\theta}(\theta_i) = T_i r^n, \quad T_i = g_{r\theta}(\theta_i), \ 1 \leqslant i \leqslant k-1
\end{aligned} \tag{59}$$

where the unknown coefficients N_i, T_i will be determined from the compatibility conditions,

$$\begin{aligned} u_r^{(i)}(r,\theta_i) &= u_r^{(i+1)}(r,\theta_i), \quad 1 \leqslant i \leqslant k-1 \\ u_\theta^{(i)}(r,\theta_i) &= u_\theta^{(i+1)}(r,\theta_i), \quad 1 \leqslant i \leqslant k-1 \end{aligned} \tag{60}$$

In Sections 2 and 3 the cartesian system of the internal and external bisector of the wedge was finally used. In the case of a multi-material wedge (Fig. 1), in order to avoid the reference in a general cartesian system, we express the displacements in the local system of the internal and external bisector of every sub-wedge. For this reason and in order to calculate the unknown coefficients N_i and T_i, the system (60) is transformed, using the relations, (22), (23), (38) and (39) for $n \leqslant -3$ or (28), (29), (44) and (45) for $n \geqslant 1$ or (56) and (57) for $n = 0$; and in the absence of normal forces relations (16) and (17) for $n = -2$, as follows

$$\begin{aligned} u_r^{(i)}\left(r, -\frac{\theta_{i-1}-\theta_i}{2}\right) &= u_r^{(i+1)}\left(r, \frac{\theta_i-\theta_{i+1}}{2}\right), \quad 1 \leqslant i \leqslant k-1 \\ u_\theta^{(i)}\left(r, -\frac{\theta_{i-1}-\theta_i}{2}\right) &= u_\theta^{(i+1)}\left(r, \frac{\theta_i-\theta_{i+1}}{2}\right), \quad 1 \leqslant i \leqslant k-1 \end{aligned} \tag{61}$$

where

$$\theta_0 = \alpha \quad \text{and} \quad \theta_k = \beta.$$

After the determination of N_i and T_i, we may calculate the stress field of every sub-wedge in terms of the local cartesian system of the internal and external bisector.

6. Applications

First consider the case of a bi-material isotropic wedge (Fig. 5). The sub-wedge (1) with mechanical properties (κ_1, μ_1) occupies the sector with angle $\theta_1(= -\alpha)$ while the sub-wedge (2), with mechanical properties (κ_2, μ_2), occupies the sector with angle $\theta_2(= \beta)$. The external faces of the bi-material wedge are

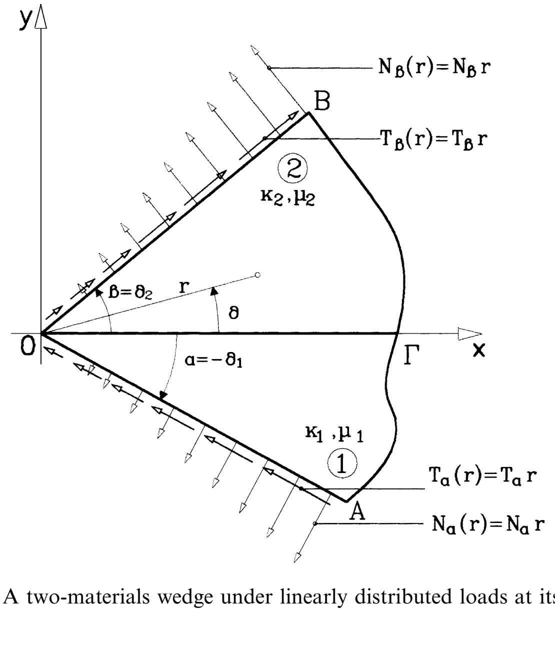

Fig. 5. A two-materials wedge under linearly distributed loads at its faces.

loaded with distributed shear and normal loads of the form Cr (order of r, $n = 1$). Taking into consideration the compatibility of displacements along the interface $O\Gamma$, from relations (60) it is obtained

$$u_i^{(1)}(r,0) = u_i^{(2)}(r,0), \quad i = r,\theta \tag{62}$$

and the unknown distributions of normal and shear stresses along the interface ($O\Gamma$), are given by

$$\begin{aligned} N_\gamma(r) &= N_\gamma r; \quad N_\gamma = \frac{F_R\Theta_T + F_\Theta R_T}{R_N\Theta_T + \Theta_N R_T} \\ T_\gamma(r) &= T_\gamma r; \quad T_\gamma = \frac{F_\Theta R_N - F_R R_N}{R_N\Theta_T + \Theta_N R_T} \end{aligned} \tag{63}$$

in case that

$$R_N\Theta_T + \Theta_N R_T \neq 0 \tag{64}$$

where

$$\begin{aligned} F_R &= \frac{\mu_2(\kappa_1+1)}{2\sin\theta_1}\left(-\frac{N_\alpha}{\tan\theta_1} + T_\alpha\right) + \frac{\mu_1(\kappa_2+1)}{2\sin\theta_2}\left(\frac{N_\beta}{\tan\theta_2} + T_\beta\right) \\ F_\Theta &= \frac{\mu_2(\kappa_1+1)}{2\sin\theta_1\tan\theta_1}\left(N_\alpha\frac{2-\tan^2\theta_1}{\tan\theta_1} - 3T_\alpha\right) + \frac{\mu_1(\kappa_2+1)}{2\sin\theta_2\tan\theta_2}\left(N_\beta\frac{2-\tan^2\theta_2}{\tan\theta_2} + 3T_\beta\right) \\ R_N &= \frac{1}{2}\left\{\mu_2\left[(\kappa_1-3) - \frac{\kappa_1+1}{\tan^2\theta_1}\right] - \mu_1\left[(\kappa_2-3) - \frac{\kappa_2+1}{\tan^2\theta_2}\right]\right\} \\ R_T &= \frac{\mu_2(\kappa_1+1)}{\tan\theta_1} + \frac{\mu_1(\kappa_2+1)}{\tan\theta_2} \\ \Theta_N &= \frac{\mu_2(\kappa_1+1)}{\tan^3\theta_1} + \frac{\mu_1(\kappa_2+1)}{\tan^3\theta_2} \\ \Theta_T &= \frac{1}{2}\left\{\mu_2\left[\frac{3(\kappa_1+1)}{\tan^2\theta_1} + (\kappa_1+5)\right] - \mu_1\left[\frac{3(\kappa_2+1)}{\tan^2\theta_2} + (\kappa_2+5)\right]\right\} \end{aligned} \tag{65}$$

Using relations (27)–(29) for the shear loading and relations (43)–(45) for the normal loading, the stress and displacements fields may be obtained for any r.

As a second application consider a three-materials isotropic wedge with a parabolic distributed shear and normal loading (order of r, $n = 2$) along its external faces (Fig. 6). The sub-wedge (1) occupies the sector $\theta_1 \leqslant \theta \leqslant \beta$ and has mechanical properties (κ_1, μ_1). The sub-wedge (2) occupies the sector $\theta_2 \leqslant \theta \leqslant \theta_1$ with mechanical properties (κ_2, μ_2) while the sub-wedge (3) occupies the sector $a \leqslant \theta \leqslant \theta_2$ with mechanical properties (κ_3, μ_3). In the case of the three-materials wedge the system (61) is written

$$\begin{aligned} u_r^{(1)}\left(r, -\frac{\beta-\theta_1}{2}\right) &= u_r^{(2)}\left(r, \frac{\theta_1-\theta_2}{2}\right) \\ u_\theta^{(1)}\left(r, -\frac{\beta-\theta_1}{2}\right) &= u_\theta^{(2)}\left(r, \frac{\theta_1-\theta_2}{2}\right) \\ u_r^{(2)}\left(r, -\frac{\theta_1-\theta_2}{2}\right) &= u_r^{(3)}\left(r, \frac{\theta_2-a}{2}\right) \\ u_\theta^{(2)}\left(r, -\frac{\theta_1-\theta_2}{2}\right) &= u_\theta^{(3)}\left(r, \frac{\theta_2-a}{2}\right) \end{aligned} \tag{66}$$

For the solution of the above system the coefficients Γ_{21}, Γ'_{21}, Γ_{42} and Γ'_{42}, are determined from relations (29) and (45) taking $n = 2$ for every sub-wedge, thus

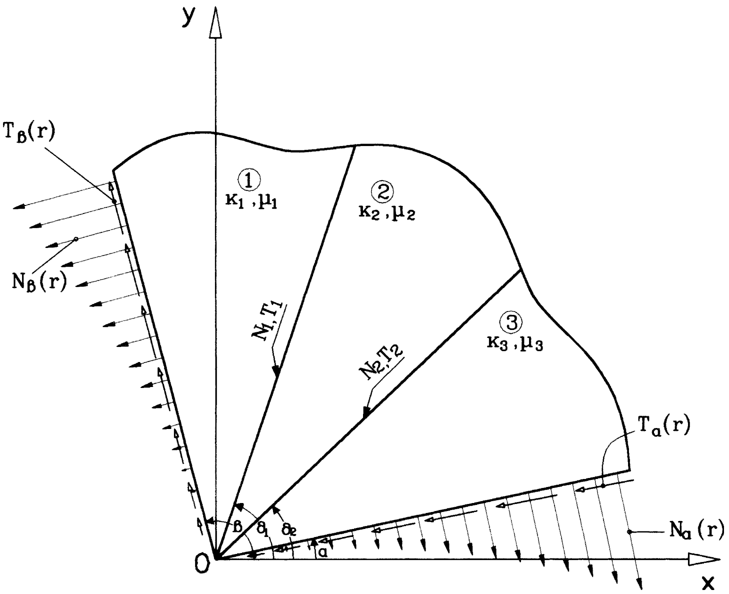

Fig. 6. A three-material wedge under parabolically distributed loads at its faces.

$$
\begin{aligned}
\Gamma_{21}^{(j)} &= \frac{T_j - T_{j-1}}{6}\beta_{1j} + \frac{N_j + N_{j-1}}{6}\gamma_{1j}, \quad j = 1,2,3 \\
\Gamma_{21}^{\prime(j)} &= \frac{T_j + T_{j-1}}{6}\beta_{2j} - \frac{N_j - N_{j-1}}{6}\gamma_{2j}, \quad j = 1,2,3 \\
\Gamma_{42}^{(j)} &= -\frac{T_j - T_{j-1}}{6}\beta_{3j} - \frac{N_j + N_{j-1}}{12}\gamma_{3j}, \quad j = 1,2,3 \\
\Gamma_{42}^{\prime(j)} &= -\frac{T_j + T_{j-1}}{6}\beta_{4j} + \frac{N_j - N_{j-1}}{12}\gamma_{4j}, \quad j = 1,2,3
\end{aligned}
\tag{67}
$$

where

$$
\begin{aligned}
\beta_{1j} &= \frac{\cos[2(\theta_{j-1} - \theta_j)]}{\sin[3(\theta_{j-1} - \theta_j)] + 3\sin(\theta_{j-1} - \theta_j)}, \quad j = 1,2,3 \\
\gamma_{1j} &= \frac{\sin[2(\theta_{j-1} - \theta_j)]}{\sin[3(\theta_{j-1} - \theta_j)] + 3\sin(\theta_{j-1} - \theta_j)}, \quad j = 1,2,3 \\
\beta_{2j} &= \frac{\sin[2(\theta_{j-1} - \theta_j)]}{\sin[3(\theta_{j-1} - \theta_j)] - 3\sin(\theta_{j-1} - \theta_j)}, \quad j = 1,2,3 \\
\gamma_{2j} &= \frac{\cos[2(\theta_{j-1} - \theta_j)]}{\sin[3(\theta_{j-1} - \theta_j)] - 3\sin(\theta_{j-1} - \theta_j)}, \quad j = 1,2,3 \\
\beta_{3j} &= \frac{\cos(\theta_{j-1} - \theta_j)}{\sin[3(\theta_{j-1} - \theta_j)] + 3\sin(\theta_{j-1} - \theta_j)}, \quad j = 1,2,3 \\
\gamma_{3j} &= \frac{\sin(\theta_{j-1} - \theta_j)}{\sin[3(\theta_{j-1} - \theta_j)] + 3\sin(\theta_{j-1} - \theta_j)}, \quad j = 1,2,3 \\
\beta_{4j} &= \frac{\sin(\theta_{j-1} - \theta_j)}{\sin[3(\theta_{j-1} - \theta_j)] - 3\sin(\theta_{j-1} - \theta_j)}, \quad j = 1,2,3 \\
\gamma_{4j} &= \frac{\cos(\theta_{j-1} - \theta_j)}{\sin[3(\theta_{j-1} - \theta_j)] - 3\sin(\theta_{j-1} - \theta_j)}, \quad j = 1,2,3
\end{aligned}
\tag{68}
$$

with

$$T_0 = T_\beta, \quad N_0 = N_\beta, \quad T_3 = T_\alpha, \quad N_3 = N_\alpha$$

and

$$\theta_0 = \beta, \quad \theta_3 = a$$

Taking into consideration relations (67) and (68), the compatibility conditions (66) take the form

$$\begin{aligned} &\frac{(\kappa_j - 3)}{\mu_j}\left[\Gamma_{21}^{(j)}\cos(\theta_{j-1}-\theta_j) - \Gamma_{21}'^{(j)}\sin(\theta_{j-1}-\theta_j)\right] - \frac{4}{\mu_j}\left\{\Gamma_{42}^{(j)}\cos[2(\theta_{j-1}-\theta_j)] - \Gamma_{42}'^{(j)}\sin[2(\theta_{j-1}-\theta_j)]\right\} \\ &\quad = \frac{(\kappa_{j+1}-3)}{\mu_{j+1}}\left[\Gamma_{21}^{(j+1)}\cos(\theta_j-\theta_{j+1}) + \Gamma_{21}'^{(j+1)}\sin(\theta_j-\theta_{j+1})\right] - \frac{4}{\mu_{j+1}}\left\{\Gamma_{42}^{(j+1)}\cos[2(\theta_j-\theta_{j+1})] \right. \\ &\qquad \left. + \Gamma_{42}'^{(j+1)}\sin[2(\theta_j-\theta_{j+1})]\right\}, \quad j = 1,2 \end{aligned} \tag{69a}$$

$$\begin{aligned} &-\frac{(\kappa_j + 3)}{\mu_j}\left[\Gamma_{21}^{(j)}\sin(\theta_{j-1}-\theta_j) + \Gamma_{21}'^{(j)}\cos(\theta_{j-1}-\theta_j)\right] - \frac{4}{\mu_j}\left\{\Gamma_{42}^{(j)}\sin[2(\theta_{j-1}-\theta_j)] + \Gamma_{42}'^{(j)}\cos[2(\theta_{j-1}-\theta_j)]\right\} \\ &\quad = \frac{\kappa_{j+1}+3}{\mu_{j+1}}\left[\Gamma_{21}^{(j+1)}\sin(\theta_j-\theta_{j+1}) - \Gamma_{21}'^{(j+1)}\cos(\theta_j-\theta_{j+1})\right] + \frac{4}{\mu_{j+1}}\left\{\Gamma_{42}^{(j+1)}\sin[2(\theta_j-\theta_{j+1})] \right. \\ &\qquad \left. - \Gamma_{42}'^{(j+1)}\cos[2(\theta_j-\theta_{j+1})]\right\}, \quad j = 1,2 \end{aligned} \tag{69b}$$

From relations (69a) and (69b) a linear system of equation is derived with unknowns N_1, N_2, T_1 and T_2. The existence of solutions of the above system if its determinant is different from zero, ensures the variable-separable form of the multi-material wedge stress field.

In a plane strain case if the angles at the apex and the mechanical properties of a three-materials wedge, are

$$\begin{aligned} &\beta - \theta_1 = \pi/6, \quad \kappa_1 = 1.60, \quad \mu_1 = 26 \text{ GPa for the sub-wedge (1)} \\ &\theta_1 - \theta_2 = \pi/6, \quad \kappa_2 = 1.88, \quad \mu_2 = 39 \text{ GPa for the sub-wedge (2)} \\ &\theta_2 - \alpha = \pi/6, \quad \kappa_3 = 1.76, \quad \mu_3 = 80 \text{ GPa for the sub-wedge (3)} \end{aligned}$$

then the linear system of (69a) and (69b), takes the form

$$\begin{aligned} &0.903T_1 - 0.154N_1 + 0.256T_2 - 0.29N_2 = -0.346T_\beta - 0.400N_\beta \\ &0.156T_1 - 0.452N_1 - 0.295T_2 + 0.128N_2 = -0.400T_\beta - 0.173N_\beta \\ &0.128T_1 + 0.148N_1 + 0.281T_2 - 0.0854N_2 = -0.060T_\alpha + 0.069N_\alpha \\ &0.295T_1 + 0.128N_1 + 0.203T_2 - 0.281N_2 = 0.138T_\alpha - 0.060N_\alpha \end{aligned} \tag{70}$$

and for

$$T_\alpha = -1, \quad N_\alpha = T_\beta = N_\beta = 1$$

the system (70) has the solution

$$T_1 = -0.243, \quad N_1 = 1.397, \quad T_2 = 0.210, \quad N_2 = 1.238 \tag{71}$$

The stress and the displacement fields in the local system of every sub-wedge, are

$$\begin{aligned}\sigma_{rr}^{(1)}(r,\theta) &= r^2[-0.384\cos(4\theta) - 0.828\sin(4\theta)]\\ \sigma_{r\theta}^{(1)}(r,\theta) &= r^2[0.582\sin(2\theta) + 0.912\cos(2\theta) + 0.384\sin(4\theta) - 0.828\cos(4\theta)]\\ \sigma_{\theta\theta}^{(1)}(r,\theta) &= r^2[1.164\cos(2\theta) - 1.824\sin(2\theta) + 0.384\cos(4\theta) + 0.828\sin(4\theta)]\end{aligned} \tag{72}$$

$$\begin{aligned}u_r^{(1)}(r,\theta) &= \frac{r^3}{52}[-0.136\cos(2\theta) + 0.213\sin(2\theta) - 0.128\cos(4\theta) - 0.276\sin(4\theta)]\\ u_\theta^{(1)}(r,\theta) &= \frac{r^3}{52}[0.446\sin(2\theta) + 0.699\cos(2\theta) + 0.128\sin(4\theta) - 0.276\cos(4\theta)]\end{aligned} \tag{73}$$

$$\begin{aligned}\sigma_{rr}^{(2)}(r,\theta) &= r^2[0.840\cos(4\theta) - 0.204\sin(4\theta)]\\ \sigma_{r\theta}^{(2)}(r,\theta) &= r^2[1.002\sin(2\theta) + 0.102\cos(2\theta) - 0.840\sin(4\theta) - 0.204\cos(4\theta)]\\ \sigma_{\theta\theta}^{(2)}(r,\theta) &= r^2[2.004\cos(2\theta) - 0.204\sin(2\theta) - 0.840\cos(4\theta) + 0.204\sin(4\theta)]\end{aligned} \tag{74}$$

$$\begin{aligned}u_r^{(2)}(r,\theta) &= \frac{r^3}{78}[-0.187\cos(2\theta) + 0.019\sin(2\theta) + 0.280\cos(4\theta) - 0.068\sin(4\theta)]\\ u_\theta^{(2)}(r,\theta) &= \frac{r^3}{78}[0.815\sin(2\theta) + 0.083\cos(2\theta) - 0.280\sin(4\theta) - 0.068\cos(4\theta)]\end{aligned} \tag{75}$$

$$\begin{aligned}\sigma_{rr}^{(3)}(r,\theta) &= r^2[-0.396\cos(4\theta) + 1.164\sin(4\theta)]\\ \sigma_{r\theta}^{(3)}(r,\theta) &= r^2[0.534\sin(2\theta) - 1.128\cos(2\theta) + 0.396\sin(4\theta) + 1.164\cos(4\theta)]\\ \sigma_{\theta\theta}^{(3)}(r,\theta) &= r^2[1.068\cos(2\theta) + 2.256\sin(2\theta) + 0.396\cos(4\theta) - 1.164\sin(4\theta)]\end{aligned} \tag{76}$$

$$\begin{aligned}u_r^{(3)}(r,\theta) &= \frac{r^3}{160}[-0.110\cos(2\theta) - 0.233\sin(2\theta) - 0.132\cos(4\theta) + 0.388\sin(4\theta)]\\ u_\theta^{(3)}(r,\theta) &= \frac{r^3}{160}[0.424\sin(2\theta) - 0.895\cos(2\theta) + 0.132\sin(4\theta) + 0.388\cos(4\theta)]\end{aligned} \tag{77}$$

The stress and displacement fields for the three-materials isotropic wedge are given in Figs. 7–11 for $r = 1$.

7. Discussion and conclusions

The problem of the infinite multi-material isotropic wedge is studied in linear elasticity under normal and shear distributed loading at its faces. The solution of the problem is proved to be self-similar and is expressed in a variable-separable form because of the geometry of the wedge and the form of the loading. In the present work, a procedure based on the self-similarity property is developed for the analysis of the infinite multi-material wedge, which allows any kind of linear elastic isotropic materials to be considered. A computer code has been implemented for the calculation and the graphical representation of the displacement and stress fields in any sub-wedge of the multi-material wedge. The tool developed has been used to analyze the corner configurations that appear in a typical metal to metal or metal to composite joint under the action of external distributed loads.

The proposed analysis gives the stress and displacement fields for mononymous load-distributions in terms of the radius r of the polar coordinate system and for different values of the order n of r.

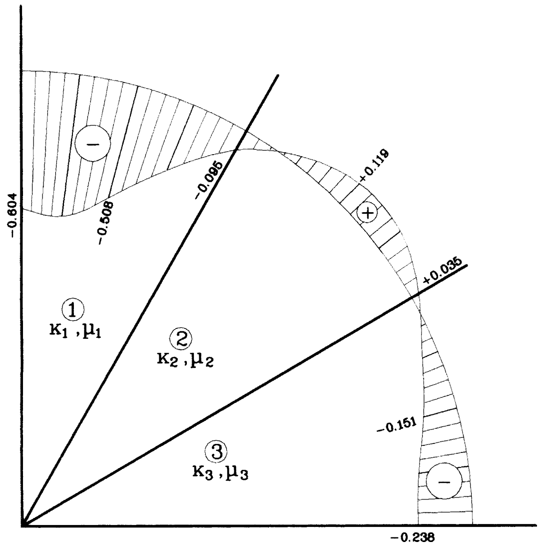

Fig. 7. $100u_r$-diagram of the three-materials wedge for $r = 1$.

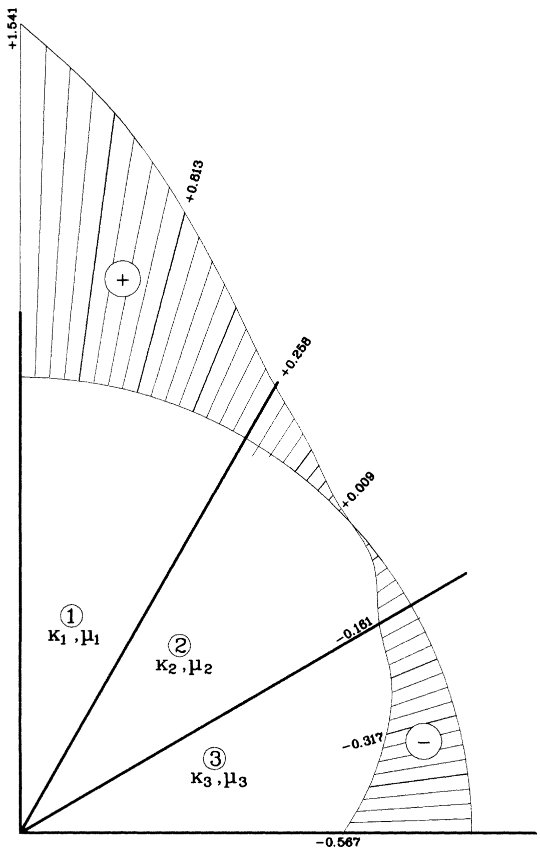

Fig. 8. $100u_\theta$-diagram of the three-materials wedge for $r = 1$.

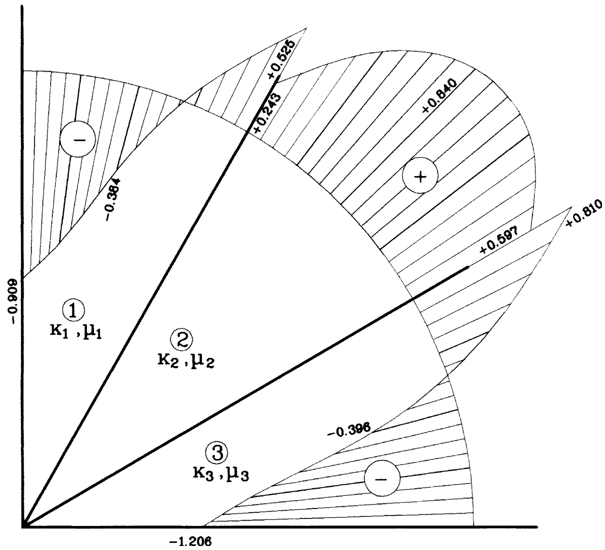

Fig. 9. σ_{rr}-diagram of the three-materials wedge for $r = 1$.

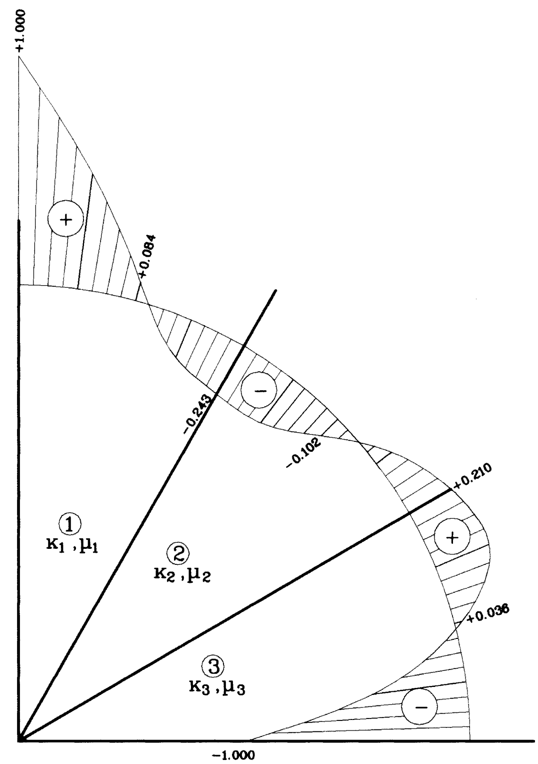

Fig. 10. $\sigma_{r\theta}$-diagram of the three-materials wedge for $r = 1$.

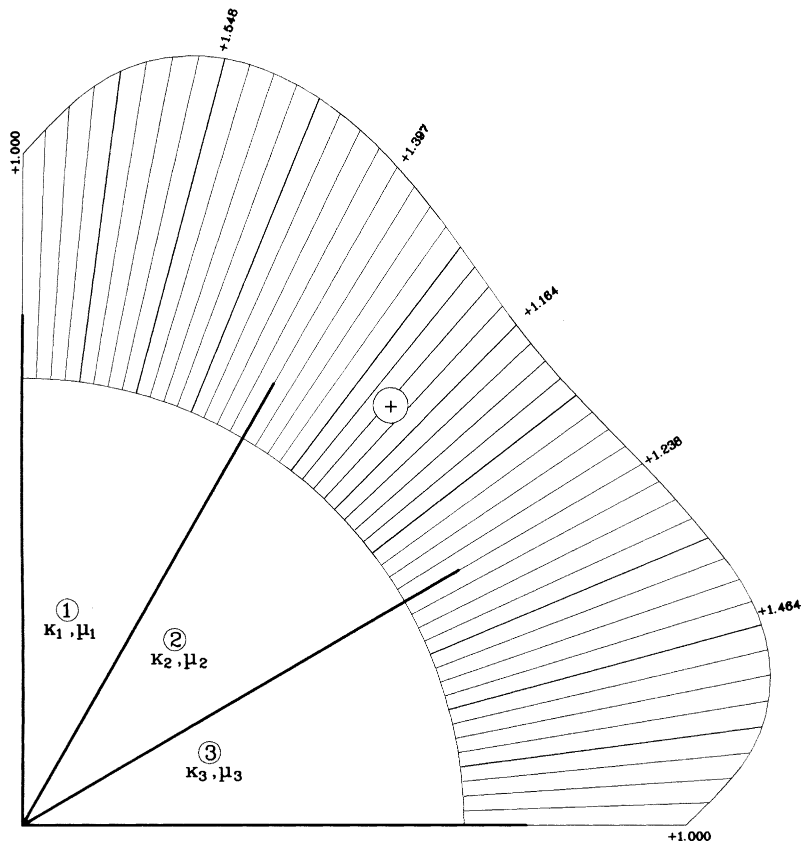

Fig. 11. $\sigma_{\theta\theta}$-diagram of the three-materials wedge for $r = 1$.

A multi-material wedge under a polynomial distributed loading along its external faces may be studied with the proposed theory using the superposition principle for every monomial-type of loading. The proposed study does not investigate the type and the order of singularity at the vertex of a multi-material wedge under normal and shear loading at its faces. It concerns the whole area of the wedge except for the region at the neighborhood of the singular point. We focus on the possibility of determining the stress and displacements fields in any plane multi-material wedge in the absence of body forces, using the superposition principle for different loading cases fulfilling a variable-separable (self-similar) solution.

From Figs. 7 and 8, continuity of the displacement field along the interfaces is observed as was expected from the theory (Section 5). From Figs. 10 and 11, continuity of the $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ stress-fields along the interfaces is also observed whereas for the σ_{rr} stress-field (Fig. 9), a discontinuity appears at the interfaces due to different material properties. The variable-separable relations (self-similarity property) are applied to any case of material combinations and to any case of order of r for the distributed loads at the external faces (except the below mentioned special cases) while restrictions for the material properties are required in the case of a multi-material wedge loaded by a concentrated force at its apex, in order to have a variable-separable formulation (Pageau et al., 1994; Joseph and Zhang, 1998).

Our investigation in the special cases of the order of n ($n = -1$ for shear and normal loading and $n = -2$ for normal loading only) in a bi-material wedge and the study of a sandwich three-materials wedge, will be the subjects of our future research.

References

Barber, J.R., 1992. Elasticity. Kluwer Academic Publishers, The Netherlands.

Bogy, D.B., 1968. Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading. ASME J. Appl. Mech. 35, 460–466.

Bogy, D.P., 1970. On the problem of edge-bonded elastic quarter planes loaded at the boundary. Int. J. Solids Struct. 6, 1287–1313.

Chen, M.P., 1998. Stress singularities in anisotropic multi-material wedges and junctions. Int. J. Solids Struct. 35, 1057–1073.

Dempsey, J.P., Sinclair, G.B., 1981. On the singular behaviour of a bi-material wedge. J. Elasticity 11, 317–327.

Dundurs, J., 1969. Discussion. ASME J. Appl. Mech. 36, 650–652.

Dundurs, J., Markenscoff, X., 1989. The Sternberg–Koiter conclusion and other anomalies of the concentrated couple. ASME J. Appl. Mech. 56, 240–245.

Gdoutos, E.E., Theocaris, P.S., 1975. Stress concentrations at the apex of a plane indenter acting on an elastic half plane. ASME J. Appl. Mech. 42, 688–692.

Horgan, C.O., 1998. Saint-Venant end effects for sandwich structures. In: Olsson, K.-A. (Ed.), Fourth International Conference on Sanwich Construction, vol. 1. EMAS Publishing, UK, pp. 191–200.

Joseph, P.F., Zhang, N., 1998. Multiple root solutions, wedge paradoxes and singular stress states that are not variable-separable. Compos. Sci. Technol. 58, 1839–1859.

Lau, C.W., Delale, F., 1988. Interfacial stress singularities at free edge of hybrid metal matrix composites. J. Eng. Mater. Technol. 110, 41–47.

Mantic, V., Paris, F., Canas, J., 1997. Stress singularities in 2D orthotropic corners. Int. J. Fracture 83, 67–90.

Michell, J.H., 1899. On the dired determination of stress in an elastic solid with application to the theory of plates. Proc. Lond. Math. Soc. 31, 100–124.

Pageau, S.S., Joseph, P.F., Biggers Jr., S.B., 1994. The order of stress singularities for bonded and disbonded three-material junctions. Int. J. Solids Struct. 31, 2979–2997.

Theocaris, P.S., Tsamasphyros, G., Andrianopoulos, N., 1979. The problem of the infinite wedge. Acta Mech. 34, 63–87.

Ting, T.C.T., 1984. The wedge subjected to tractions, a paradox re-examined. J. Elasticity 14, 235–240.

Ting, T.C.T., 1996. Anisotropic Elasticity: Theory and Applications. Oxford Science Publications, New York.

Williams, M.L., 1952. Stress singularities resulting from various boundary conditions in angular corners of plates in extension. ASME J. Appl. Mech. 19, 526–528.